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On the quantum mechanics of dissipative lagrangians

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Abstract. An appropriate path differential measure is employed for obtaining the path integral representation of the propagator of a particle with a time dependent 'mass'. The evaluations are restricted to quadratic lagrangians and the propagator for the damped harmonic oscillator is given explicitly.

On many occasions a constrained particle is dissipating energy to a surrounding many-particle system with which it is coupled. Although a proper quantum-dynamical treatment of such a particle would involve the dynamics of the particle plus the surrounding system an immense simplification of the problem can be attained by the construction of a single-particle lagrangian, which takes account of the energy absorbing process in an effective manner.

In general such (dissipative) lagrangians manifest the particle loss of energy to the environment in the form of a time prescribed coefficient of the velocity and an explicitly time-dependent potential energy. We shall use the term 'mass' for twice the coefficient of the velocity in these lagrangians.

They look like:

$$L[\boldsymbol{x}, t] = \tfrac{1}{2}\mathscr{M}(t)\dot{\boldsymbol{x}}^2 - U(\boldsymbol{x}, t). \tag{1}$$

As is well known the propagator of the corresponding Schrödinger equation,

$$-\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial t}\psi(\boldsymbol{x}, t) = \left(-\frac{\hbar^2}{2\mathscr{M}(t)}\nabla^2 + U(\boldsymbol{x}, t)\right)\psi(\boldsymbol{x}, t) \tag{2}$$

contains all the necessary quantum-dynamical information.

The propagator is formally obtained by making use of a conditional path integral of an appropriate path differential measure, which takes account of the 'mass' time dependence. We have:

$$K(\boldsymbol{x}t|\boldsymbol{x}'0) = \int_{\boldsymbol{x}(0)=\boldsymbol{x}'}^{\boldsymbol{x}(t)=\boldsymbol{x}} \exp\left(\frac{\mathrm{i}}{\hbar}\int_0^t [\tfrac{1}{2}\mathscr{M}(\tau)\dot{\boldsymbol{x}}^2(\tau) - U(\boldsymbol{x}(\tau), \tau)]\,\mathrm{d}\tau\right)\mathscr{D}[\boldsymbol{x}] \tag{3}$$

where the path differential measure is given by:

$$\mathscr{D}[\boldsymbol{x}] = \left(\frac{\mathscr{M}(0)}{2\pi\mathrm{i}\hbar\,\mathrm{d}\tau}\right)^{3/2} \prod_{0<\tau<t}\left(\frac{\mathscr{M}(\tau)}{2\pi\mathrm{i}\hbar\,\mathrm{d}\tau}\right)^{3/2}\mathrm{d}\boldsymbol{x}(\tau). \tag{4}$$

We shall exemplify the above by evaluating the propagator in two cases of interest, but to facilitate this we shall more or less state certain results relating to quadratic lagrangians.

From Feynman and Hibbs (1965) we know that the propagator for a quadratic lagrangian can be written exactly as:

$$K(\boldsymbol{x}t|\boldsymbol{x}'0) = K(\boldsymbol{0}t|\boldsymbol{0}\boldsymbol{0})\exp\left(\frac{\mathrm{i}}{\hbar}S(\boldsymbol{x}t|\boldsymbol{x}'0)\right) \tag{5}$$

where $S(\boldsymbol{x}t|\boldsymbol{x}'0)$ is the classical action of our particle along the path starting from $\boldsymbol{x}'$ at zero time and reaching $\boldsymbol{x}$ at time t.

$K(\boldsymbol{0}t|\boldsymbol{0}\boldsymbol{0})$ is the propagator for a cyclic return to the origin in time t. It is completely time dependent.

We now wish to point out how to evaluate the propagator $K(\boldsymbol{0}t|\boldsymbol{0}\boldsymbol{0})$ in the case of quadratic lagrangians.

By a procedure analogous to the one appearing in Papadopoulos (1968) it is shown that:

$$K(\boldsymbol{0}t|\boldsymbol{0}\boldsymbol{0}) = \left(\det\frac{\mathcal{M}(0)}{2\pi\mathrm{i}\hbar\mathbf{D}(t)}\right)^{1/2} \tag{6}$$

where $\mathbf{D}(t)$ is a matrix (3×3 in this case) function which obeys a certain second-order differential equation, relating to the lagrangian of the problem, and satisfies the initial conditions:

$$\mathbf{D}(0) = \mathbf{0}, \qquad \dot{\mathbf{D}}(0) = \mathbf{I} \tag{7}$$

($\mathbf{I} = 3\times 3$ unit matrix).

In the case of a lagrangian

$$L = \tfrac{1}{2}\mathcal{M}(t)\dot{\boldsymbol{x}}^2 - \tfrac{1}{2}(t)\boldsymbol{x}^2 + \boldsymbol{f}(t)\,.\,\boldsymbol{x} \tag{8}$$

the equation for the matrix $\mathbf{D}(t)$ becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{M}(t)\dot{\mathbf{D}}(t)) + \lambda^2(t)\mathbf{D}(t) = 0. \tag{9}$$

By now we have all we need for the evaluation of the propagator of the damped harmonic oscillator.

Consider the lagrangian (Havas 1957)

$$L_0 = \exp\left(\frac{t}{t_0}\right)\frac{m}{2}(\dot{\boldsymbol{x}}^2 - \Omega^2\boldsymbol{x}^2). \tag{10}$$

Upon application of the Euler–Lagrange equations to (10) we obtain the equation

$$m\ddot{x} + \frac{m}{t_0}\dot{\boldsymbol{x}} + m\Omega^2\boldsymbol{x} = \boldsymbol{0}, \tag{11}$$

which is the classical equation of motion for the damped harmonic oscillator. Comparing (8) and (10) we obtain from (9) the following equation for $\mathbf{D}$ in the case of the damped oscillator:

$$\ddot{\mathbf{D}} + \frac{1}{t_0}\dot{\mathbf{D}} + \Omega^2\mathbf{D} = \mathbf{0}. \tag{12}$$

The required $\mathbf{D}(t)$ satisfying the initial conditions (7) is obtained as:

$$\mathbf{D}(t) = \frac{\sin \Omega' t}{\Omega'} \exp\left(-\frac{t}{2t_0}\right) \mathbf{I}; \qquad \Omega' = \left[\Omega^2 - \left(\frac{1}{2t_0}\right)^2\right]^{1/2}. \tag{13}$$

To complete the evaluation we then require the classical action for the damped oscillator which can be obtained from the classical path $\boldsymbol{X}(\tau)$ (with $\boldsymbol{X}(0) = \boldsymbol{x}'$, $\boldsymbol{X}(t) = \boldsymbol{x}$) which satisfies the equation of motion (11).

For this action we use

$$S_0(\boldsymbol{x}t|\boldsymbol{x}'0) = \int_0^t \exp\left(\frac{\tau}{t_0}\right)\frac{m}{2}(\dot{\boldsymbol{X}}^2(\tau) - \Omega^2 \boldsymbol{X}^2(\tau))\,\mathrm{d}\tau. \tag{14}$$

The final result for the damped harmonic oscillator obtained upon combination of (5), (6), (13) and (14) is given by:

$$K_0(\boldsymbol{x}t|\boldsymbol{x}'0) = \left[\frac{m\Omega'}{2\pi \mathrm{i}\hbar \sin \Omega' t} \exp\left(\frac{t}{2t_0}\right)\right]^{3/2} \exp\left(\frac{\mathrm{i}}{\hbar}\frac{m}{4t_0}\left[\boldsymbol{x}'^2 - \boldsymbol{x}^2 \exp\left(\frac{t}{t_0}\right)\right]\right.$$
$$\left. + \frac{\mathrm{i}}{\hbar}\frac{m\Omega'}{2 \sin \Omega' t}\left\{\left[\boldsymbol{x}'^2 + \boldsymbol{x}^2 \exp\left(\frac{t}{t_0}\right)\right]\cos \Omega' t - 2\boldsymbol{x}' . \boldsymbol{x} \exp\left(\frac{t}{2t_0}\right)\right\}\right). \tag{15}$$

This result can also be verified by use of the Van Vleck–Pauli formula (Jones and Papadopoulos 1971).

The quantum mechanics of the damped harmonic oscillator was also studied by Bopp (1962) using a coordinate and a wavefunction transformation leading to a Schrödinger equation for the ordinary oscillator but with complex eigenvalues and essentially 'time-dependent' eigenfunctions. It is then possible for one to obtain expression (15) by transforming the propagator associated with the complex eigenvalues. However, the path integral method is direct and of more far reaching applicability.

We illustrate this by considering a new model-dissipative lagrangian for a particle in an external field

$$L_1 = \frac{m}{2}\left(1 + \frac{t}{t_0}\right)\dot{\boldsymbol{x}}^2 + \boldsymbol{f}(t) . \boldsymbol{x}. \tag{16}$$

It can be easily verified that the relaxation pattern of the velocity goes like $(1 + t/t_0)^{-1}$ in contrast to the faster exponential decay of the previous case.

Combining (5), (6), (7) and (9) utilizing the appropriate quantities from the lagrangian (16) we finally obtain the propagator associated with this lagrangian:

$$K_1(\boldsymbol{x}t|\boldsymbol{x}'0) = \left(\frac{m}{2\pi \mathrm{i}\hbar t_0 \ln(1 + t/t_0)}\right)^{3/2} \exp\left(\frac{\mathrm{i}}{\hbar \ln(1 + t/t_0)}\left\{\frac{m}{2t_0}(\boldsymbol{x} - \boldsymbol{x}')^2\right.\right.$$
$$+ \int_0^t \mathrm{d}\tau \left[\ln\left(\frac{t_0 + t}{t_0 + \tau}\right)\boldsymbol{x}' + \ln\left(1 + \frac{\tau}{t_0}\right)\boldsymbol{x}\right] . \boldsymbol{f}(\tau)$$
$$\left.\left. + \frac{t_0}{m}\int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau' \ln\left(\frac{t_0 + \tau}{t_0 + t}\right) \ln\left(1 + \frac{\tau'}{t_0}\right)\boldsymbol{f}(\tau) . \boldsymbol{f}(\tau')\right\}\right). \tag{17}$$

Evidently the lagrangian formulation of quantum mechanics in the case of constrained systems prevails over the usual eigenfunction method, which in general is not easily accessible.

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